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DISCRETE MATHEMATICS

Counting two-dimensional posets

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Abstract

The number of unlabeled 2-dimensional posets is recursively calculated. This counting makes use of the relationship between permutations and posets of dimension two.

1. Introduction

In 1988 El-Zahar and Sauer [2] proved that the number of pairwise non-isomorphic 2-dimensional posets on n elements is asymptotically $\frac{1}{2}n!$. Recently, Winkler [5] extended this result to labeled 2-dimensional posets and showed that the number of such posets on n elements is $(1+o(n))n!^2/(2\sqrt{e})$.

It is our purpose here to describe an exact counting, though recursive, of unlabeled 2-dimensional posets. This counting exploits the relationship between 2-dimensional posets and permutations. In fact, we are merely counting certain classes of permutations. From this counting we arrive at the number of prime 2-dimensional posets. Pólya's enumeration theorem [3] and results of Stanley [4] are then applied to calculate the number of unlabeled posets having dimension two.

2. Preliminaries and basic definitions

The *dimension* [1] of a partially ordered set (poset) P is the minimum number of linear extensions of P whose intersection is the ordering of P. Such linear extensions form a *realization* of P.

Let P be a 2-dimensional poset on n elements. Each linear extension L of P can be viewed as a bijection from P onto the set $[n]=\{1, 2, \ldots, n\}$, namely if

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$L=\{a_1<\cdots<a_n\}$ then $L(a_i)=i$. Assume that (L_1, L_2) is a realization of P. Let S_n denote the Symmetric group on $[n]$. Following [2], we define a permutation $\sigma(L_1, L_2)\in S_n$ by $\sigma(L_1, L_2)=L_1L_2^{-1}$. We say that $\sigma(L_1, L_2)$ corresponds to the poset P. Conversely, each $\sigma\in S_n$ corresponds to some 2-dimensional n-element poset P; namely, if $L_1=\{1<\cdots<n\}$ and $L_2=\{\sigma(1)<\cdots<\sigma(n)\}$ then (L_1, L_2) is a realization of P and $\sigma(L_1, L_2)=\sigma$. This correspondence between permutations and posets is many-to-one. In fact $\sigma(L_2, L_1)=(\sigma(L_1, L_2))^{-1}$ and, thus a 2-dimensional poset might correspond to more than one permutation.

However this correspondence behaves well when restricted to the so called prime posets. Let A be a subset of a poset P. A is called *P-autonomous* if for every $x_1, x_2\in A$ and $y\in P\setminus A$ we have:

(a) $x_1<y$ if and only if $x_2<y$,

(b) $y<x_1$ if and only if $y<x_2$.

The poset P is called *prime* if it does not contain an autonomous set A with $1<|A|<|P|$. The following proposition was proved in [2], see also [5].

Proposition 2.1. *If P is a prime 2-dimensional poset then P has a realization (L_1, L_2) which is unique up to the order of L_1 and L_2.*

Permutations corresponding to prime 2-dimensional posets can be easily characterized. Let $\sigma\in S_n$ and assume that $X\subseteq[n]$ is an interval of the form $X=\{i, i+1, \ldots, i+j\}$. We say that X is a *consecutive* set of σ if the set $\sigma(X)=\{\sigma(i), \ldots, \sigma(i+j)\}$ is an interval of $[n]$.

Proposition 2.2 ([2, 5]). *Let (L_1, L_2) be a realization of the two dimensional poset P. Then P is prime if and only if $\sigma(L_1, L_2)$ has no proper consecutive set.*

The above proposition will be used in the following sections to calculate the number of prime 2-dimensional posets. We devote the rest of this section to describing how the class of 2-dimensional posets is built up from prime 2-dimensional posets using the operations of the substitution composition and the series and parallel sums.

Let P and Q be two disjoint posets and let $x\in P$. The poset P_x^Q is obtained from P by replacing the element x by the poset Q such that for every $y\in Q$ and $z\in P\setminus\{x\}$; $z<y$ if and only if $z<x$ and $y<z$ if and only if $x<z$. This operation is called the substitution composition. Clearly, in this case, Q will be an autonomous set in P_x^Q.

Again let P and Q be two disjoint posets. Their parallel sum (or disjoint union) is the poset $P+Q$ defined on the union of their ground sets by

(1) if $x, y\in P$ and $x\leqslant y$ in P then $x\leqslant y$ in $P+Q$,

(2) if $x, y\in Q$ and $x\leqslant y$ in Q then $x\leqslant y$ in $P+Q$.

The series (or ordinal) sum $P\oplus Q$ is the poset defined on the union of their ground sets satisfying (1) and (2) above and the further condition

(3) if $x\in P$ and $y\in Q$ then $x\leqslant y$ in $P\oplus Q$.

A poset P is called $(+, \oplus)$-irreducible if it is not the series or parallel sum of two smaller posets. Clearly if P is a prime poset, $x \in P$ and Q is any poset then P_x^Q is $(+, \oplus)$-irreducible.

It is well known that the operations of series and parallel sums and substitution composition preserve the dimension. We record this fact as the following.

Proposition 2.3. *Let P and Q be two 2-dimensional posets and let $x \in P$. Then each of $P+Q$, $P \oplus Q$ and P_x^Q has dimension two.*

3. Prime permutations

We shall say that a permutation σ is a series, parallel, $(+, \oplus)$-irreducible or prime permutation if the poset corresponding to σ has the respective property. We make these definitions more precise as follows.

Let $\sigma \in S_n$ and $\tau \in S_m$. Then their sum is the permutation $\sigma \oplus \tau \in S_{n+m}$ defined by:

$$(\sigma \oplus \tau)(i) = \begin{cases} \sigma(i) & \text{if } 1 \leqslant i \leqslant n, \\ \tau(i-n)+n & \text{if } n+1 \leqslant i \leqslant n+m. \end{cases}$$

Their parallel sum is the permutation $\sigma + \tau \in S_{n+m}$ satisfying:

$$(\sigma + \tau)(i) = \begin{cases} m+\sigma(i) & \text{if } 1 \leqslant i \leqslant n, \\ \tau(i-n) & \text{if } n+1 \leqslant i \leqslant n+m. \end{cases}$$

Note that these operations satisfy $(\sigma+\tau)^{-1} = \tau^{-1} + \sigma^{-1}$ and $(\sigma \oplus \tau)^{-1} = \sigma^{-1} \oplus \tau^{-1}$.

Let $j \in [n]$. Then σ_j^τ is the permutation $\theta \in S_{n+m-1}$ defined by:

$$\theta(i) = \begin{cases} \sigma(i) & \text{if } i<j \text{ and } \sigma(i)<\sigma(j), \\ \sigma(i)+m-1 & \text{if } i<j \text{ and } \sigma(j)<\sigma(i), \\ \tau(i-j+1)+\sigma(j)-1 & \text{if } j \leqslant i \leqslant j+m-1, \\ \sigma(i-m+1) & \text{if } j+m \leqslant i \leqslant n+m-1 \text{ and } \sigma(i-m+1)<\sigma(j), \\ \sigma(i-m+1)+m-1 & \text{if } j+m \leqslant i \leqslant n+m-1 \text{ and } \sigma(j)<\sigma(i-m+1). \end{cases}$$

We say that σ_j^τ is obtained from σ by substituting τ and j. These permutations are illustrated schematically in Fig. 1.

In other words, if σ and τ correspond respectively to the posets P and Q, then $\sigma \oplus \tau$, $\sigma + \tau$ and σ_j^τ correspond respectively to the posets $P \oplus Q$, $P+Q$ and P_x^Q where $x \in P$ is the element corresponding to $j \in [n]$.

A permutation θ is called a series or parallel permutation if it has respectively the form $\sigma \oplus \tau$ or $\sigma + \tau$ for some σ and τ. Otherwise θ is $(+, \oplus)$-irreducible. Furthermore, θ is said to be prime if it does not have the form σ_j^τ for some σ, τ and j.

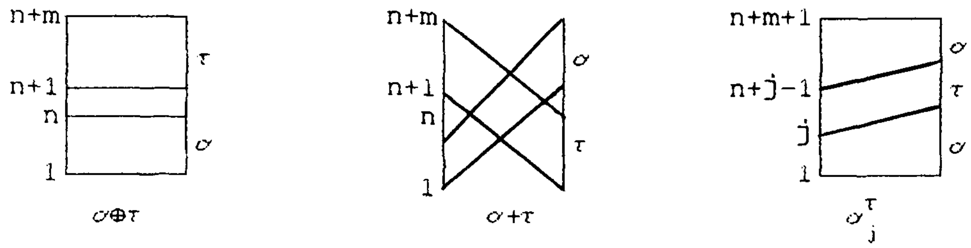

Fig. 1.

We introduce the following generating functions:

$$F^*(x)=\sum_{n\geq 1} f_n^* x^n,$$

$$V^*(x)=\sum_{n\geq 1} v_n^* x^n,$$

$$U^*(x)=\sum_{n\geq 1} u_n^* x^n,$$

$$I^*(x)=\sum_{n\geq 1} i_n^* x^n,$$

$$P^*(x)=\sum_{n\geq 1} P_n^* x^n.$$

These are respectively the generating functions for all permutations, series, parallel, $(+,\oplus)$-irreducible and prime permutations. Of course we have

$$f_n^*=n! \tag{3.1}$$

Lemma 3.1. (a) $F^*(x)=V^*(x)+U^*(x)+I^*(x)$,
(b) $V^*(x)=(F^*(x)-V^*(x))F^*(x)$,
(c) $U^*(x)=(F^*(x)-U^*(x))F^*(x)$,
(d) $V^*(x)=U^*(x)$.

Proof. (a) is clear. Now, let θ be a series permutation. We can write $\theta=\theta_1\oplus\theta_2\oplus\cdots\oplus\theta_k$ where each θ_i is a non-series permutation. Moreover this representation is unique. Thus θ can be written in a unique way as $\theta=\sigma\oplus\tau$ where τ is some permutation and σ is a non-series permutation. The generating function for non-series permutation is $F^*(x)-V^*(x)$. This proves (b). In a similar way we can prove (c). Finally (d) follows from (b) and (c). □

Lemma 3.2. $I^*(x)=x+P^*(F^*(x))$.

Proof. Let $\sigma\in S_n$ be a prime permutation. The generating function for all permutations obtained from σ by the substitution operation is $(F^*(x))^n$ since every such permutation has the form $\sigma_{1,\ldots,n}^{\tau_1,\ldots,\tau_n}$ where $\tau_1,\tau_2,\ldots,\tau_n$ are arbitrary permutations. From this the lemma follows. □

Lemma 3.1 together with Eq. (3.1) can be used to calculate recursively v_n^*, u_n^* and i_n^*. Then using Lemma 3.2 we can calculate P_n^*.

Recall that an involution is a self-inverse permutation. Let $\sigma \in S_n$ be an involution. An element $i \in [n]$ is a *fixed* element of σ if $\sigma(i)=i$. A pair of distinct elements $i, j \in [n]$ is a conjugate pair if $\sigma(i)=j$ (and $\sigma(j)=i$).

Let $F^*(y,z)=\sum_{j,k\geqslant 0} f_{j,k}^* y^j z^{2k}$ where $f_{j,k}^*$ is the number of involutions $\sigma \in S_{j+2k}$ having j fixed elements and k pairs of conjugate elements. Then $f_{j,k}^*$ is the number of permutations having the cycle type $1^j\, 2^k$. Hence

$$f_{j,k}^* = \frac{(j+2k)!}{j!\ k!\, 2^k}. \tag{3.2}$$

Let us further introduce the generating functions:

$$V^*(y,z)=\sum_{j,k\geqslant 0} v_{j,k}^* y^j z^{2k},$$

$$U^*(y,z)=\sum_{j,k\geqslant 0} u_{j,k}^* y^j z^{2k},$$

$$I^*(y,z)=\sum_{j,k\geqslant 0} i_{j,k}^* y^j z^{2k},$$

$$P^*(y,z)=\sum_{j,k\geqslant 0} P_{j,k}^* y^j z^{2k},$$

where $v_{j,k}^*$, $u_{j,k}^*$, $i_{j,k}^*$ and $P_{j,k}^*$ are respectively the number of series, parallel, $(+,\oplus)$-irreducible and prime involutions of cycle type $1^j\, 2^k$.

Clearly we have

$$F^*(y,z)=V^*(y,z)+U^*(y,z)+I^*(y,z). \tag{3.3}$$

Lemma 3.3. $V^*(y,z)=(F^*(y,z)-V^*(y,z))\,F^*(y,z)$.

Proof. Similar to part (b) of Lemma 3.1. □

Lemma 3.4. $U^*(y,z)=(F^*(z^2)-U^*(z^2))(1+F^*(y,z))$.

Proof. Let θ be a parallel involution. Write $\theta=\theta_1+\theta_2+\cdots+\theta_k$ where each θ_i is a non-parallel permutation. Since $\theta^{-1}=\theta_k^{-1}+\cdots+\theta_2^{-1}+\theta_1^{-1}$. Let $\sigma=\theta_1$ and $\tau=\theta_2+\cdots+\theta_{k-1}$ (for $k>2$). Thus θ can be written in either the form $\sigma+\tau+\sigma^{-1}$ (if $k>2$) or the form $\sigma+\sigma^{-1}$ (if $k=2$) where σ is a non-parallel permutation and τ is an involution. The absence of τ is accounted for by the term 1 in the second bracket. □

Lemma 3.5. $I^*(y,z)=P^*(F^*(y,z),(F^*(z^2))^{1/2})$.

Proof. Let σ be a prime involution with cycle type $1^j 2^k$. Consider any involution θ obtained from σ by the substitution operation. Then θ is constructed from σ by substituting an involution for fixed elements and a pair τ and τ^{-1} for some permutation τ for each pair of conjugate elements. Therefore, the generating function for all involutions obtained by substitution from σ is $(F^*(y,z))^j(F^*(z^2))^k$. The lemma now follows. □

Equations (3.2) and (3.3) and Lemmas 3.3, 3.4 and 3.5 can be used to calculate recursively $V^*(y,z)$, $U^*(y,z)$, $I^*(y,z)$ and $P^*(y,z)$. In the following section we shall use $P^*(x)$ and $P^*(y,z)$ to calculate the number of unlabeled 2-dimensional posets.

4. Two-dimensional posets

Let us introduce some more generating functions. Let

$$F(x)=\sum_{n\geq 1} f_n x^n,$$

$$V(x)=\sum_{n\geq 1} v_n x^n,$$

$$U(x)=\sum_{n\geq 1} u_n x^n,$$

$$I(x)=\sum_{n\geq 1} i_n x^n,$$

$$P(x)=\sum_{n\geq 1} P_n x^n$$

where f_n, v_n, u_n, i_n and P_n denote respectively the number of total, series, parallel, $(+,\oplus)$-irreducible and prime unlabeled 2-dimensional posets with n elements.

Theorem 4.1. $P(x)=\frac{1}{2}(P^*(x)-P^*(x,x))+P^*(x,x)$.

Proof. Let σ be a prime permutation. From Proposition 2.2, the poset corresponding to σ is then prime. Note that both σ and σ^{-1} correspond to the same poset. The theorem now follows since $P^*(x,x)$ and $P^*(x)-P^*(x,x)$ are the generating functions for prime involutions and prime permutations which are not involutions. □

Lemma 4.2. *Let P be a prime 2-dimensional poset and denote by $\Gamma(P)$ its automorphism group. Then* $|\Gamma(P)|\leq 2$.

Proof. Let L_1, L_2 be the unique linear extensions of P which form a realization of P. Assume that $\tau\in\Gamma(P)$. Then $(L_1\tau,\ L_2\tau)$ is also a realization of P. Therefore either $L_1\tau=L_1$ and $L_2\tau=L_2$ or $L_1\tau=L_2$ and $L_2\tau=L_1$. In the former case τ is the identity and in the latter case τ is an involution of P. This shows that $\Gamma(P)$ have at most two elements. □

Let P be a prime 2-dimensional poset. Assume that P has a non-trivial automorphism group $\Gamma(P)$, say $\Gamma(P)=\{\mathrm{id}, \tau\}$. An element $x \in P$ for which $\tau(x)=x$ will be called a *fixed element* and a pair of distinct elements x, y for which $\tau(x)=y$ and $\tau(y)=x$ will be called *conjugate*. If P has the trivial automorphism group then, by convention, all elements of P are considered to be fixed.

Let $P(y,z)=\sum_{j,k\geqslant 0} P_{j,k}\, y^j z^{2k}$ where $P_{j,k}$ is the number of prime unlabeled 2-dimensional posets having j fixed elements and k pairs of conjugate elements. We can re-state Theorem 4.1 as the following.

Theorem 4.3. $P(y,z)=\frac{1}{2}(P^*(y)-P^*(y,y))+P^*(y,z)$.

Lemma 4.4. $I(x)=x+\frac{1}{2}\sum_{j,k\geqslant 0} P_{j,k}(F^{j+2k}(x)+F^j(x)\,F^k(x^2))$.

Proof. Let P be a prime 2-dimensional poset with j fixed elements and k pairs of conjugate ones. Then the automorphism group, $\Gamma(P)$, of P will have the cycle index

$$Z(\Gamma(P))=\tfrac{1}{2}(s_1^{j+2k}+s_1^j s_2^k).$$

From Pólya's enumeration theorem (see for example [2, Ch. 3, p. 35]), the generating function for all 2-dimensional posets obtained from P by the substitution composition is

$$\tfrac{1}{2}(F^{j+2k}(x)+F^j(x)\,F^k(x^2)).$$

The term x on the right-hand side accounts for the fact that the single-element poset is $(+,\oplus)$-irreducible but cannot be obtained by substitution from a prime poset. □

Lemma 4.4 concludes the recursive calculation of $I(x)$, the generating function for $(+,\oplus)$-irreducible 2-dimensional posets. The class of 2-dimensional posets is the set of all posets obtained from the class of $(+,\oplus)$-irreducible 2-dimensional posets by the operations of disjoint union and linear sum. Stanley [4] describes how to count the class of all posets obtainable from a given class of $(+,\oplus)$-irreducible posets by the operations of linear sum and disjoint union. This result of Stanley allows us to obtain the following relations between the functions $F(x)$, $V(x)$, $U(x)$ and $I(x)$.

Proposition 4.5. (a) $F(x)=V(x)+U(x)+I(x)$.

(b) $V(x)=(F(x)-V(x))\,F(x)$.

(c) $1+F(x)=\exp(\sum_{k\geqslant 1}(V(x^k)+I(x^k))/k)$.

The above proposition completes the recursive calculation of f_n, the number of unlabeled 2-dimensional posets with n elements. The results of these calculations for $n\leqslant 15$ are included in the appendix.

Appendix

$P_{j,k}$: the number of prime unlabeled 2-dimensional posets having j fixed elements and k pairs of conjugate elements.
P_n: the number of prime unlabeled 2-dimensional posets with n elements.
i_n: the number of $(+, \oplus)$-irreducible unlabeled 2-dimensional posets with n elements.
v_n the number of series unlabeled 2-dimensional posets with n elements.
u_n: the number of parallel unlabeled 2-dimensional posets with n elements.
f_n: the total number of unlabeled 2-dimensional posets with n elements.

Table 1
$P_{j,k}$

j \ k	0	1	2	3	4	5	6	7
0	0	0	0	1	8	83	1003	13935
1	0	0	2	9	90	1027	13967	217727
2	0	0	3	28	351	5064	82790	
3	0	0	1	36	713	13976	287122	
4	1	0	0	20	825	24048		
5	2	0	0	4	543	26784		
6	21	0	0	0	189			
7	164	0	0	0	27			
8	1445	0	0	0				
9	14010	0	0	0				
10	149036	0	0					
11	1726345	0	0					
12	21639817	0						
13	291903582	0						
14	4216933310							
15	64970340763							

Table 2

n	P_n	i_n	v_n	u_n	f_n
1	0	1	0	0	1
2	0	0	1	1	2
3	0	0	3	2	5
4	1	1	9	6	16
5	4	12	32	19	63
6	25	101	134	80	315
7	174	876	688	392	1956
8	1481	8105	4294	2395	14794
9	14136	81678	32258	17590	131526
10	149490	895498	283894	152456	1331848
11	1728089	10637749	2847100	1512186	14997035
12	21646709	136202070	31803211	16762666	184767947
13	291932068	1870476606	389315584	204045690	2463837890
14	4217054272	27434128174	5164688162	2965183977	35294000313
15	64970872423	428132304745	73671613533	38311550328	540115468606

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