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Asymptotic Enumeration of N-Free Partial Orders

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Abstract. Let $N(n)$ and $N^*(n)$ denote, respectively, the number of unlabeled and labeled N-free posets with n elements. It is proved that $N(n) = 2^{n \log n + o(n \log n)}$ and $N^*(n) = 2^{2n \log n + o(n \log n)}$. This is obtained by considering the class of N-free interval posets which can be easily counted.

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1. Introduction

Let $(P, \leqslant)$ be a partially ordered set (poset), i.e. a nonempty set P together with a reflexive, antisymmetric, and transitive binary relation $\leqslant$ on P. For short, $(P, \leqslant)$ will be denoted by its ground set P. The poset P is called *N-free* if its directed covering graph has no induced subgraph isomorphic to the digraph N shown in Figure 1.

The class of N-free posets was first introduced by P. Grillet [4]. In [9], I. Rival introduced the term N-free.

Another class related to N-free posets is the class of *series-parallel* posets, i.e. posets which can be obtained from the single-element poset by series and parallel composition. It is known [13] that P is series-parallel if and only if every induced subposet of P is N-free and, hence, the class of series-parallel posets is a subclass of the N-free posets class. This result was also independently proved by Kaerekes and Möhring [6]. The smallest poset which is N-free but not series-parallel is the poset with five elements illustrated in Figure 2.

Let $S(n)$ and $N(n)$ denote, respectively, the number of unlabeled series-parallel and N-free posets with n elements. R. Stanley [11] used the technique of generating functions to calculate $S(n)$ and gave the estimate $S(n) \sim Cn^{-3/2}\alpha^{-n}$ for some constants C and α, which gives a lower bound for $N(n)$. In [10, p. 525], R. Möhring asked about the relative frequency of series-parallel posets within the class of N-free posets. On the other hand, Habib and Möhring [5] combined with Kleitman and

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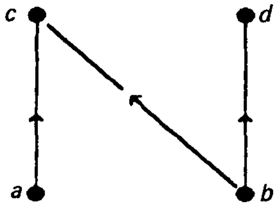


Fig. 1.

Rothschild's estimate [7] for the number of partial orders, to show that almost all posets are not N-free.

The purpose of this paper is to prove that: $N(n) = 2^{n \log n + o(n \log n)}$. (All logarithms have the base 2.) Comparing this value with the result of Stanley [11], one concludes that almost all N-free posets are not series-parallel, which answers the question of Möhring.

2. Asymptotic Estimate of $N(n)$

Let P be a finite N-free poset. By a *block* of P we mean a maximal complete bipartite graph in the directed covering graph of P. More precisely, a block of P has the form (A, B), where $A, B \subseteq P$ are such that A is the set of all upper covers (in P) of every $y \in B$ and B is the set of all lower covers of every $x \in A$. By convention, (Min P, $\emptyset$) and ($\emptyset$, Max P) are also blocks where Min P and Max P are the minimal and maximal elements of P.

Let $(A_1, B_1), \ldots, (A_k, B_k)$ be all the blocks of P. Note that for any two elements $x, y \in P$, the sets of lower covers of x and y are either disjoint or identical. The same is true for the sets of upper covers. Thus, the A_i's from a partition of P and so do the B_i's. We shall always assume that the blocks of P are ordered such that for any $x \in P$ if $x \in A_i$ and $x \in B_j$ then $i < j$. We get the block representation of P by filling a $2 \times k$ array with the A_i's in the first row and the B_i's in the second row in the above order. This is illustrated in Figure 3. Clearly, every N-free poset has a unique block representation apart from a possible permutation of the columns of the array. Then we can get an upper bound on $N(n)$ by bounding the number of blocks with n elements.

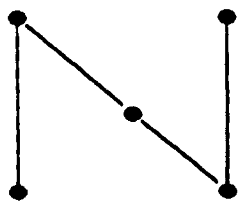
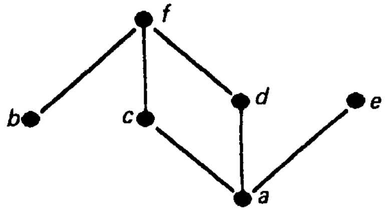


Fig. 2.



a, b	c, d, e	f	∅
∅	a	b, c, d	e, f

Fig. 3.

PROPOSITION 1. $N(n) \leqslant 2^{n \log n + o(n \log n)}$.

Proof. Let P be an N-free poset with n elements and let $(A_1, B_1), \ldots, (A_k, B_k)$ denote the blocks of P ordered as stated above. Define

$$a_i = |A_i| \quad \text{and} \quad b_i = |B_i|, \quad \text{for } i = 1, \ldots, k.$$

Denote the elements of P by $u_1, \ldots, u_n$, where

$$A_1 = \{u_1, \ldots, u_{a_1}\}, \qquad A_2 = \{u_{a_1+1}, \ldots, u_{a_1+a_2}\}, \ldots,$$

and so on.

Thus, the first row of the block representation of P is completely determined by the composition (i.e. partition into parts whose order counts) $n = a_1 + a_2 + \cdots + a_{k-1}$ of n into $k-1$ positive parts. Let $u_{\sigma(1)}, \ldots, u_{\sigma(n)}$ denote a permutation of the elements of P such that

$$B_2 = \{u_{\sigma(1)}, \ldots, u_{\sigma(b_2)}\}, \qquad B_3 = \{u_{\sigma(b_2+1)}, \ldots, u_{\sigma(b_2+b_3)}\}, \ldots,$$

and so on. Then the second row of the block representation is completely determined by the composition $n = b_2 + \cdots + b_k$ and the permutation σ. Since the number of compositions of n is 2^{n-1}, then we get

$$N(n) \leqslant 2^{n-1} \cdot 2^{n-1} \cdot n! = 2^{n \log n + o(n \log n)}.$$

This completes the proof of the proposition. □

In order to prove the lower bound on $N(n)$, we exhibit a class of N-free posets of size $2^{n \log n + o(n \log n)}$. As it turns out, this class will consist of posets which are simultaneously N-free and *interval order*. Recall that a poset is an interval order if it does not contain two parallel edges, i.e. an induced subposet of four elements a, b, c, d with $a < b$ and $c < d$ (the only comparabilities), see Figure 4.

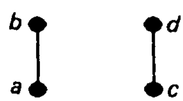


Fig. 4.

Now, let P be an N-free poset with n elements and k blocks $(A_1, B_1), \ldots, (A_k, B_k)$ ordered as before. Define a $k \times k$ matrix

$$M(P) = [m_{ij}], \quad \text{where } m_{ij} = |A_i \cap B_j|.$$

The prescribed order of the blocks implies that $m_{ij} = 0$ whenever $i \geqslant j$, that is $M(P)$ has zeros on and below the main diagonal. Again, $M(P)$ is unique up to a possible permutation σ applied simultaneously to the rows and the columns. The following matrix is an illustration of $M(P)$, where P is that of Figure 3.

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

LEMMA 2. *Assume that* $m_{i,i+1} \neq 0$ *for* $i = 1, \ldots, k-1$. *Then* $M(P)$ *is unique. Moreover, if* $m_{ij} \leqslant 1$ *for all* i, j, *then* P *is rigid, i.e., has no nontrivial automorphism.*

Proof. Since $m_{i,i+1} \neq 0$ we have $A_i \cap B_{i+1} \neq \emptyset$, and then the ith block must precede the $(i+1)$st block in any block representation of P. This is true for all i, hence P has a unique block representation and, consequently, $M(P)$ is unique.

Now, assume further that all $m_{ij} \leqslant 1$. Let x_i be the unique element of $A_i \cap B_{i+1}$ for $i = 1, \ldots, k-1$. Then $x_1 < x_2 < \cdots < x_{k-1}$ is a unique maximum chain of length $k-2$ in P. Suppose α is an automorphism of P. Then $\alpha(x_i) = x_i$ for each $i \in \{1, \ldots, k-1\}$. It remains to prove that α fixes every other element of P. Let $x \in P$, say $\{x\} = A_i \cap B_j$ for $j > i+1$. Then x is the unique element of P which covers x_{i-1} (or minimal for $i = 1$) and is covered by x_j (or maximal for $j = k$). Therefore, $\alpha(x) = x$. This shows that P is rigid, which completes the proof of Lemma 2. □

LEMMA 3. *Let* P *and* $M(P)$ *be as above. Then* P *is an interval order if and only if* $m_{i,i+1} \neq 0$ *for all* $i = 1, \ldots, k-1$.

Proof. Assume that $A_i \cap B_{i+1} = \emptyset$ for some i. Choose an edge $a < b$ from the ith block (A_i, B_i) and an edge $c < d$ from the $(i+1)$st block (A_{i+1}, B_{i+1}). These two edges are then parallel.

Conversely, let P contain two parallel edges $a < b$ and $c < d$. We can assume that these edges are covering edges, say the edge $a < b$ belongs to (A_i, B_i) and the edge $c < d$ belongs to (A_j, B_j) where $i < j$. Suppose there are elements

$$x_h \in A_h \cap B_{h+1} \quad \text{for } i \leqslant h \leqslant j-1.$$

Then $a < x_i < \cdots < x_{j-1} < d$ which is a contradiction, and the proof of Lemma 3 is complete. □

Let $J(x) = \Sigma_{n \geqslant 1} j(n) x^n$ be the generating function of all N-free interval orders in which no two distinct elements have the same lower covers and the same upper covers.

LEMMA 4.

$$j(n) = \sum_{k=2}^{n+1} \binom{(k-1)(k-2)/2}{n-k+1}.$$

Proof. Let P be an N-free interval poset with n elements and k blocks, and assume that no two elements of P have simultaneously the same lower covers and the same upper covers. Then $M(P) = [m_{ij}]$ is a unique $0-1$ matrix in which all the $(i, i+1)$ entries are 1's. Thus, the value of m_{ij}, $j \leqslant i+1$ is independent of P. The remaining entries of $M(P)$ can be chosen in

$$\binom{(k-1)(k-2)/2}{n-k+1}$$

ways. Summing over k, we get the required result. □

Let $I(x) = \Sigma_{n \geqslant 1} i(n)x^n$ be the generating function of all N-free interval posets. Replacing an element in a poset with an antichain produces a set of elements with the same lower covers and the same upper covers. The generating function of all antichains is $x/(1-x)$. Therefore

$$I(x) = J\left(\frac{x}{1-x}\right).$$

Now we complete the proof of our main results.

THEOREM 5. $N(n) = 2^{n \log n + o(n \log n)}$.

Proof. Since $N(n) \geqslant j(n)$, then it is sufficient to estimate $j(n)$. Assume n is large and put $m = n/\log n + 2$. Now lemma 4 implies that

$$j(n) > \binom{m^2/2}{n-m+1}.$$

Using Stirling's formula, one easily deduces that if $a \gg b \gg 1$, then

$$\log\binom{a}{b} \sim b \log\frac{a}{b}.$$

Therefore

$$\log j(n) \sim (n-m) \log\frac{m^2}{n-m} \sim n \log n$$

which completes the proof of the theorem. □

Let $N^*(n)$ denote the number of labeled N-free posets with n elements.

THEOREM 6. $N^*(n) = 2^{2n \log n + o(n \log n)}$.

Proof. The posets counted by $j(n)$ are rigid, hence there are $n!$ ways to label the elements of such a poset. Therefore

$$N^*(n) \geqslant 2^{2n \log n + o(n \log n)}.$$

On the other hand

$$N^*(n) \leqslant n!N(n) = 2^{2n \log n + o(n \log n)}.$$

Table I.

n	CN	N	CSP	SP	NI	I
1	1	1	1	1	1	1
2	1	2	1	2	2	2
3	3	5	3	5	5	5
4	9	15	9	15	14	15
5	31	49	30	48	43	53
6	115	180	103	167	143	217
7	474	715	375	602	510	1014
8	2097	3081	1400	2256	1936	5335
9	9967	14217	5380	8660	7775	31240
10	50315	69905	21073	33958	32869	201608
11	268442	363926	83950	135292	145665	1422074
12	1505463	1996922	338878	546422	674338	10886503

N: N-free posets; CN: connected N; SP: series-parallel posets; CSP: connected SP; NI: N-free interval posets; I: interval posets.

Appendix

In this appendix, we present the number of unlabeled N-free, series-parallel posets with n elements $n \leqslant 12$. The calculation of the number of series-parallel posets is based on [11]. The number of N-free posets was calculated through a computer program which generates all matrices $M(P)$ representing N-free posets P taking into account that different matrices may represent the same poset. The number of matrices increased rapidly with n, so did the running time and the calculations had to be stopped at $n = 12$ (Table I). For comparison, we also include the number of unlabeled N-free interval posets and the number of unlabeled interval posets (based on [3]) for $n \leqslant 12$.

Finally, let us remark that several of the numbers of partial order with $n = 10$ elements given by Möhring in [8] are not correct; compare the numbers of N-free and interval orders above. See also [1] for the exact number of two-dimensional posets with 10 elements.

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