THE 25th ANNUAL CONFERENCE ON STATISTICS, COMPUTER SCIENCE AND OPERATIONS RESEARCH

COUNTING PERMUTATION GRAPHS

BY

BAYOUMI I. BAYOUMI MOHAMED H. EL-ZAHAR
AND
SOHEIR M. KHAMIS

ORGANIZED BY
Institute of Statistical Studies and Research
Cairo University
EGYPT
8-10 December 1990

The 25th Annual Conference
I.S.S.R, Cairo University, Vol. 25, 1990

COUNTING PERMUTATION GRAPHS

*Bayoumi I. Bayoumi**, *Mohamed H.* EL-Zahar*, and *Soheir M.* Khamis.*

Abstract.

In this paper the authors enumerate the number of permutation graphs (up to isomorphism). The counting process depends on the exploitation of structural decompositions of various sorts of permutations and permutation graphs and a relationship between them, which leads to the properties of those graphs having twice transitive orientations (prime permutation graphs).

According to the results of Pσlya, Riddell (see [4] ch. 2,4), the exact number of specified graphs. g(n), with n vertices is, recursively, calculated. Also, the authors found the functional equation for the generating function $\sum_{n\geq 1} g(n)x^n$ in terms of the generating function for prime permutation graphs.

The numbers of permutation graphs with $n \leq 20$ are given. These numbers show that the previously known number for $n = 10$ given in [6] is not correct.

AMS Mathematics Subject Classification (1980). 05A15 and 06A10.

Keywords.
Permutation, comparability graph, permutation graph, partitive set, prime graph, prime permutation graph, transitive orientation, poset of dimension two, generating function.

*Department of Mathematics, Faculty of Science,
Ain Shams University, Cairo, Egypt.

§1 . Introduction.

Previously known studies about permutation graphs have been based on two aspects of these graphs, namely their theoretical structure and the design of algorithms for this class. In recent years the structure of a graph can be represented by a tree. The tree representation s for permutation graphs, [7],have contained the infinite class of prime permutation graphs as basic building blocks. In [8]Lorna k. Stewart develope4 a decomposition theorem for prime permutation graphs and designed a new permutation graph tree structure which has only the paths on fewer than four points and their complements as basic building blocks.

Recently, in [2], the authors have enumerated 2-dimensional posets together with many sorts of permutations. Here a 2-dimensional counting technique is extended to describe exact counting, through recursion, of permutation graphs (up to isomorphism). The technique depends on the correspondence between the permutations and the permutation graphs which is many-to-one, since two or more permutations may have the same comparability graph.

In fact, the authors are merely counting five disjoint sub-classes of permutations. which lead to the number of prime permutation graphs, then they applied Pσlya's and Riddell's theorems. [4], to calculate the number of unlabeled permutation graphs.

The appendix contains all computational results for some types of permutation graphs and prime permutation graphs of $n \leq 20$ points.

§2. Definitions and Basic Concepts.

Henceforth, the term "graph" will be used for unlabeled, simple, undirected graph unless otherwise stated. Let G = (V.E) be a *graph* and $\bar{G}$ is its *complement*. A graph can be *oriented* into a directed graph by transforming each undirected edge (x,y) ∈ E into a directed edge (x,y) or (y,x).

A directed graph is *transitive* if the existence of the edges (x,y) and (y,z) in E implies the existence of (x,z) in E. If there are three vertices x, y and z such that (x,y) and (y,z) are edges but (x,z) is not an edge, there is a *transitvity violation* involving x, y and z. A directed graph is transitive iff there are no transitivity violations between any trio of vertices.

A graph is a *comparability graph* if it can be oriented into a transitive graph. Therefore, the *comparability graph*, G(P), of a partially ordered set (poset), P, is the graph G=(V.E), where V = P and any two vertices x,y are adjacent in G iff they are comparable in P, (i.e. x < y or y < x).

A comparability graph G is said to be a *uniquely partially orderable graph*, or a *UPO*, if it has exactly two transitive orientations, one being the reverse of the other, ([1], [5]). A proper subfamily of UPOs is the family of *UTOs*, where a comparability graph G is said to be a *uniquely two-dimensional orientable graph* or a *UTO*, if both G and $\bar{G}$ are UPOs.

A graph G is a *permutation graph*, [7], if there is a pair of permutations P_1, P_2 on the vertex set, such that there is an edge between u and v in G iff u precedes v in P_1 and P_2, or v precedes u in P_1 and P_2. Every permutation graph is a comparability graph; the converse is not true. Also, *A graph G is a permutation graph iff G and* $\bar{G}$ *are comparability graphs*, [3].

According to this definition, one can conclude, [8], that:

Permutation graphs are precisely the comparability graphs of two-dimensional posets. By the *dimension, d(P),* of a poset P, we mean the minimum number of linear extensions of P whose intersection is the ordering of P, and the set of 2-dimensional posets consists of all posets having $d(P) \leq 2$, [2].

This shows that permutation graphs and 2-dimensional posets can be identified with a permutation on the ground set, [2]. This characterization [6] can be obtained by renumbering the elements of a 2-dimensional poset in such a way that one of the two linear extensions whose intersection is the ordering of P is just $v_1 v_2 \ldots v_n$. The other linear extension $v_{i_1} v_{i_2} \ldots v_{i_n}$ then defines the permutation π by letting $\pi(v_j) = v_{i_j}$. Thus a graph G = (V,E) with $V = \{v_1, \ldots, v_n\}$ is a permutation graph iff there is a permutation π on V with

$$(v_i, v_j) \in E \text{ iff } i < j \text{ and } \pi^{-1}(v_i) < \pi^{-1}(v_j),$$

where $\pi^{-1}(v_i)$ denotes the position of v_i in π. An example of a permutation graph is given in Fig.1, that identified with the permutation $\pi = \{v_6 v_3 v_7 v_1 v_5 v_2 v_4\}$ on its ground set $\{v_1 v_2 v_3 v_4 v_5 v_6 v_7\}$.

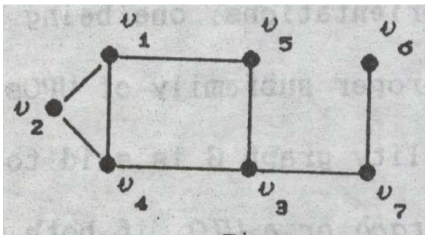


Fig.1.

As in the case of 2-dimensional posets [2], a permutation graph might correspond to more than one permutation. Therefore, the correspondence between permutations and permutation graphs is many-to- one. However, this correspondence behave well when restricted to the so called prime permutation graphs. Since there are only a finite number of permutation of each permutation graph.

For any graph G =(V,E), a subset $U \subset V$ is said to be a *partitive On G* if every vertex in V-U is adjacent to all vertices or to vertex in U. The graph G is said to be *prime* iff it has no *ntps* (non-trivial partitive set). We know that, every prime comparability graph is a UPO [8]and also the prime permutation graphs together with the path of three vertices and its complement are precisely the UTOs [8]. Therefore, one can easily conclude that : Every *prims permutation graph is* a *UTO.*

Permutations corresponding to prime permutation graphs can be easily characterized by knowing the form of a representation of a partitive set of a graph G in its corresponding permutation. Let $\sigma \in S_n$ and assume that $X \subseteq [n]$ is an interval of the form $X = \{i, i+1, \ldots, i+j\}$. We say that X is a *consecutive* set of σ if the set $\sigma(X) = \{\sigma(i), \sigma(i+1), \ldots, \sigma(i+j)\}$ is an interval of [n]. From [8], if $\sigma \in S_n$ be the corresponding permutation to the permutation graph $G = (V, E)$ with $|V| = n$, then s *has* a *consecutive set if G has* a *ntps.* In other words *G is* a *prime iff s has* no *proper* consecutive *set* [2].

This result enables us (in the next section) to calculate the prime permutation graphs. Now, we introduce how to build up the class of permutation graphs (permutations) from prime permutation graphs (prime permutations) by using substitution composition , series sum and parallel sum.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two disjoint graphs. Their Parallel sum *(disjoint* union) $G_1 \cup G_2$ is defined by the ordered pair $(V_1 \cup V_2, E_1 \cup E_2)$,and the *series* sum (sum), $G_1 + G_2$ is obtained by adding all edges between G_1 and G_2 to $G_1 \cup G_2$.

Let $G = (V, E)$ and g , (V', E') be two disjoint graphs. The graph G_v^g is obtained from G by replacing the vertex $v \in V$ by the graph g through replacing every edge meets v in E by a new edge meets y, ∀ y ∈ V'. This operation is called substitution composition. Clearly, in this case, V' will be a partitive set in G_v^g. A graph G is called *(∪,+)-irreducible* if it is not the series or parallel sum of two smaller graphs. Obviously, if G is a prime graph, v ∈ V and g is any graph then G_v^g is (∪,+)-irreducible graph.

It is well-known that the operations of substitution composition, series sum and parallel sum preserve the property of being permutation graph if the initial graphs are permutation graphs,[8].

The construction principle for the composite of permutation graphs carries over their corresponding permutations. Let permutations σ,τ and π correspond to the graphs G_1,G_2 and D respectively. Then, the union sum $G_1 \cup G_2$ corresponds to the parallel composition $\sigma+\tau$, whereas the series sum $G_1 + G_2$ corresponds to the series composition $\sigma \oplus \tau$. And also δ_j^∂ corresponds to the graph $D_v^{G_1}$ where $v \in D$ is the vertex corresponding to $j \in [n]$. (The authors introduced the definitions of these operations in case of permutations in more details in [2]). For simplicity, henceforth the two graph operations, disjoint sum and series sum, will also be denoted by "+" and "⊕" respectively. Note that the parallel and series sums for permutations are not commutative in the contrast to the case for graphs. Fig.2 is a schematically example of the permutation operations.

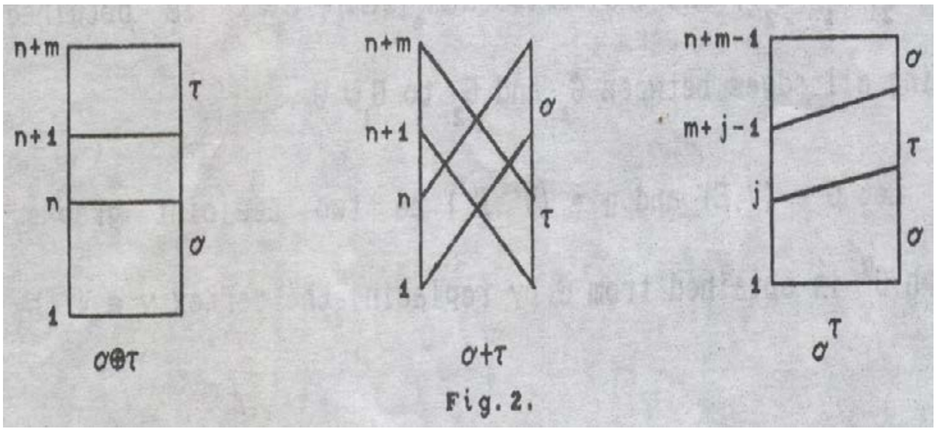


Fig.2.

Thus a permutation δ is called *series* or *parallel* permutation if it has the form $\sigma \oplus \tau$ or $\sigma + \tau$ for some σ and τ respectively. otherwise δ is *$(+,\oplus)$-irreducible* Furthermore δ is said to be prime if it does not have the form σ_j^{τ} for some σ, τ and j.

In the remainder of the present section, we introduce the various types of symmetries of permutations. These types together with the above properties of prime permutation graphs lead to some facts which simplify the counting problem of the exact number of permutation graphs.

Let $\sigma(G)$ denotes the set of all permutations which represent a permutation graph G. Let PG(σ) denote the unique graph represented by a permutation σ.

Permutations which represent the same graph are equivalent in a sense. We now introduce a more restricted type of relationship which partitions $\sigma(G)$ into equivalence classes of cardinality less than or equal to four. Let σ be a permutation on [n]. Then the permutation obtained by flipping σ vertically (the dual permutation) is referred to as σ^v, the permutation obtained by flipping σ horizontally (the inverse permutation) is referred to as σ^h and the permutation obtained by flipping σ both vertically and horizontally is σ^{vh}. We now define

$\Gamma(\sigma) = \{\sigma, \sigma^v, \sigma^h, \sigma^{vh}\}$. For any permutation σ, all elements of $\Gamma(\sigma)$ certainly represent PG(σ). Further characterizations of the elements of $\Gamma(\sigma)$ are introduced in the following lemma, [8].

Lemma 2.1 .

(1) $\sigma^v(i) = n + 1 - \sigma(n+1-i) \quad 1 \le i \le n.$

(2) $\sigma^h = \sigma^{-1}$.

(3) $\sigma^{vh}(i)$=n+1$-\sigma^{-1}$ (n+l-i) $1\leq i \leq n$. □

It is clear that for a permutation σ, the elements of $\Gamma(\sigma)$ are not necessarily all distinct. Fig.3 illustrates examples of several types of symmetries. According to the following types, we can divide the class of permutation into five subclasses.

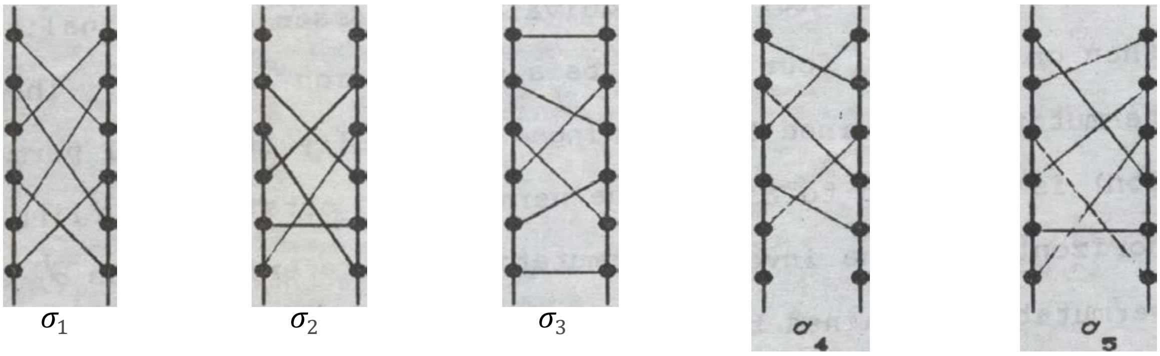
Type (1) : a permutation σ has all possible symmetries, that is $\sigma = \sigma^{v} = \sigma^{h} = \sigma^{vh}$ (σ_1 in Fig.3) and $|\Gamma(\sigma)| = 1$.

Type (2) : a Permutation σ has only $\sigma = \sigma^{h}$ ($\sigma^{v} = \sigma^{vh}$) but $\sigma \neq \sigma^{v}$ and $\sigma \neq \sigma^{vh}$ (σ_2 in Fig.3). So, $|\Gamma(\sigma)| = 2$, and σ is called an *involution*.

Type (3) :a permutation σ has only $\sigma = \sigma^{v}$ ($\sigma^{h} = \sigma^{vh}$) but $\sigma \neq \sigma^{h}$ and $\sigma \neq \sigma^{vh}$ (σ_3 in Fig.3). So, $|\Gamma(\sigma)| = 2$, and σ is called a *self dual* permutation.

Type (4) : a permutation σ has only $\sigma = \sigma^{vh}$ ($\sigma^{h} = \sigma^{v}$) but $\sigma \neq \sigma^{h}$ and $\sigma \neq \sigma^{v}$ (σ_4 in Fig.3).so, $|\Gamma(\sigma)| = 2$.

Type (5) :a permutation σ has none of the symmetries, therefore $|\Gamma(\sigma)| = 4$, (σ_5 in Fig.3).



σ_1 σ_2 σ_3 σ_4 σ_5

Fig.3.

In the following theorem Lorna,[8], proved that two different labellings which give rise to the same permutation imply the existence of an automorphism in the graph.

Theorem 2.2.

Let σ and τ be two permutations representing the graph G=(V,E) and let $\{v_1, v_2, ..., v_n\}$ and $\{u_1, u_1, ..., u_n\}$ be the labellings of V corresponding to σ and τ, respectively. Then $\sigma = \tau$ iff there is an automorphism of G mapping $\{v_1, v_2, ..., v_n\}$ onto $\{u_1, u_1, ..., u_n\}$. □

Now we restrict our attention to prime permutation graphs and their corresponding permutations. We exploit the following lemma, [8], concerning the number of permutations that represent the same UTos.

Lemma 2.3.

Let G be a permutation graph and let $\sigma \in \sigma(G)$. Then G is a UTO iff the only permutations for G are $\Gamma(\sigma)$, i.e. $|\sigma(G)| = |\Gamma(\sigma)|$ □.

Now, we can translate this information into the language of generating functions.

§3. Generating Functions For Types of Prime Permutations.

In the rest of the paper, standard generating function techniques are employed to first count the number of prime permutation graphs and then to count, for each prime permutation graph, the number of $(+,\oplus)$-irreducible permutation graphs. According to series, parallel sums and Riddel's theorem, the number of permutation graphs is obtained. Our first job will be to enumerate the symmetric classes of permutations. To do so, we first recall the results that proved in [2] for obtaining the generating functions for permutations and involutions.

The following are the generating functions for all, series, parallel, $(+,\oplus)$-irreducible and prime permutations respectively.

$F(x) = \Sigma_{n\geq 1} f(n)x^n$, $V(x) = \Sigma_{n\geq 1} v(n)x^n$, $U(x) = \Sigma_{n\geq 1} u(n)x^n$,

$I(x) = \Sigma_{n\geq 1} i(n)x^n$ and $P(x) = \Sigma_{n\geq 1} P(n)x^n$.

It is clear that $f(n) = n!$ and the needed relations between these generating functions are given below, [2].

$$F(x) = 2V(x) + I(X) \qquad (3.1).$$

(Note that the generating function for parallel permutations is equal to the generating function for series permutations, [2]).

$$V(x) = (F(x) - V(x))F(x) \qquad (3.2).$$

$$I(X) = P(F(x)) + x \qquad (3.3).$$

These relations can be used to calculate v(n} and i(n) recursively. Then using (3.3) one can calculate P(n).

The elements of permutation can be divided into sets depending on whether they fall in a 1-cycle, or 2-cycle when the permutation is flipped. To obtain the generating functions for symmetric classes of permutations, these sets must be determined and therefore, the following terms must be defined.

Let $\sigma \in S_n$ be an involution (a self dual). Then an element $i \in [n]$ is a *fixed* element of σ if $\sigma(i) = i$ (if n is an odd number, i is the middle point and $\sigma(i)=i$). A pair of distinct elements $i, j \in [n]$ is a *conjugate pair* of symmetric points if $\sigma(i) = j$ and $\sigma(j) = i$ ($\sigma(i) + \sigma(j) = i + j = n + 1$).

The class of permutations can split into two parts, one contains all involutions ($\sigma = \sigma^h$) and the other contains the rest of specified class ($\sigma \neq \sigma^h$). This partition is already enumerated in [2], thus we recall only the considered generating functions and relations between them.

Let $F_1(y,z) = \Sigma_{j,k\geq 0} f_1(j,k) y^j z^{2k}$ where $f_1(j,k)$ is the number of involutions $\partial \in S_{j+2k}$ having j fixed elements and k pairs of conjugate elements. Then $f_1(j,k)$ is the number of permutations having the cycle type $1^j 2^k$. Hence

$$f_1(j,k) = \frac{(j+2k)!}{j!\ k!\ 2^k} \qquad (3.4).$$

From [2] we have

$V_1(y,z) = \Sigma_{j,k\geq 0}\, v_1(j,k)\, y^j z^{2k}, U_1(y,z) = \Sigma_{j,k\geq 0}\, u_1(j,k)\, y^j z^{2k},$
$I_1(y,z) = \Sigma_{j,k\geq 0} i_1(j,k)\, y^j z^{2k} \,\&\, P_1(y,z) = \Sigma_{j,k\geq 0}\, P_1(j,k)\, y^j z^{2k},$

where $v_1(j,k)$, $u_1(j,k)$, $i_1(j,k)$ and $P_1(j,k)$ are the number of series, parallel, $(+,\oplus)$-irreducible and prime involutions of cycle type $1^j 2^k$ respectively. Obviously, we have

$$F_1(y,z) = V_1(y,z) + U_1(y,z) + I_1(y,z) \qquad (3.5).$$

Furthermore the following relations between these generating functions are proved in [2].

$$V_1(y,z) = (F_1(y,z) - V_1(y,z))F_1(y,z) \qquad (3.6).$$

$$U_1(y,z) = (F(z^2) - U(z^2))(1 + F_1(y,z)) \qquad (3.7).$$

$$I_1(y,z) = P(F_1(y,z), (F(z^2))^{1/2}) \qquad (3.8).$$

Equations (3.4)-(3.8) can be used to calculate recursively the coefficients of $V_1(y,z)$, $U_1(y,z)$, $I_1(y,z)$ and $P_1(y,z)$.

Again, we can divide the class of permutation w.r.t. the flipping on vertically. A similar process will be used to obtain the generating functions for self dual permutations in which $\sigma=\sigma^v$.

Let $F_2(y,z) = \Sigma_{k\geq 0} f_2(k)(1+y)z^{2k}$ where $f_2(k)$ is the number of self dual permutations $\sigma \in S_n$ Sn having the cycle type 2^k where $k=\lfloor n/2 \rfloor$. Hence

$$f_2(k) = 2^k\ k! \qquad (3.9).$$

Consider $V_2(y,z)$, $U_2(y,z)$, $I_2(y,z)$ and $P_2(y,z)$ be the generating function for series, parallel, $(+,\oplus)$-irreducible and prime self dual permutations of cycle type 2^k respectively. It is easy to see that these are related by equation (3.5). In what follows the authors proved some more relations between these generating functions.

Lemma 3.1.

(a) $V_2(y,z) = (F(z^2) - V(z^2))(1 + F_2(y,z))$.

(b) $U_2(y,z) = (F(z^2) - U(z^2))(1 + F_2(y,z))$.

(c) $U_2(y,z) = V_2(y,z)$.

Proof.

Let θ be a series self dual permutation. Then θ can be written uniquely in either the form $\partial \oplus \tau \oplus \partial$ or the form $\sigma \oplus \sigma$ where σ is non-series permutation and τ is a self dual permutation because of $\theta = \theta^v$. The absence of τ is accounted for by the term 1 in the second bracket. This proves (a). By a similar way we can prove (b).

Finally (c) follows from (a) and (b). □

So, equation (3.5) can be rewritten for this type as.

$$F_2(y,z) = 2V_2(y,z) + I_2(y,z) \quad (3.10).$$

Lemma 3.2.

$$I_2(y,z) = P_2(F_2(y,z), (F(z^2))^{1/2}).$$

Proof.

Let $\sigma \in S_n$ be a prime self dual permutations with cycle type 2^k, where k = $\lfloor n/2 \rfloor$. Consider any self dual permutation θ obtained from σ by substituting a self dual permutation for the middle point if n is odd and the same permutation for each pair of conjugate elements. Therefore, the generating function for all self dual permutations obtained by substitution from σ is $F_2(y,z)(F(z^2))^k$.
The lemma now follows. □

Equations (3.9) and (3.10) and lemmas 3.1 and 3.2 can be used to calculate recursively $V_2(y,z)$, $I_2(y,z)$ and $P_2(y,z)$.

The class of permutations can be split again into two halves according to the condition $\sigma^h = \sigma^v$. From lemma 3.1, if $\sigma \in S_n$ and.

$\sigma^h = \sigma^v$, we get

$$\sigma(i) = n+1-\sigma^h(n+1-i) \quad \forall\ i \in [n] \qquad (3.11).$$

This rule partitions the elements of $[n]$ into two parts. Elements of the first part satisfy $\sigma(i) = n+1-i$. The second part contains all elements $\iota, k \in [n]$ satisfying $\sigma(\iota) = n+1-k$ (and $\sigma(k) = n+1-\iota$). Thus elements ι, k are also called conjugate elements. Now we get the generating function for these permutations.

Fortunately, it is good surprise, that the class of permutations σ having $\sigma^v = \sigma^h$ can be easily turned out to the class of involutions by using the following transformation on the R.H.S. of equation (3.11).

$$\sigma'(i) = n+1-\sigma(i) = \sigma^h(i).$$

Therefore, $\sigma'(i) = i$ if $\sigma(i) = n+1-i$ otherwise if ι, k are conjugate elements then $\sigma'(k) = \iota$ and $\sigma'(\iota) = k$. This means that σ' is an involution, for example see Fig.4. This transformation leads to the following fact.

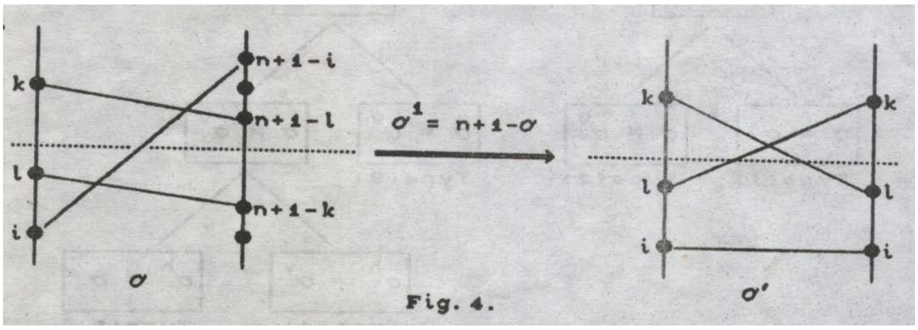


Fig. 4.

Proposition 3.3.

There exist a 1-1 correspondence between the class of self inverse permutations (involutions) and the class of permutations σ having $\sigma^v = \sigma^h$. □

So, it is enough to consider the generating functions, $I_3(y,z)$ and $P_3(y,z)$ for all, series, $F_3(y,z)$, $V_3(y,z)$, $U_3(y,z)$, parallel, $(+,\oplus)$-irreducible and prime permutations with $\sigma^h = \sigma^v$

respectively. These generating functions functions are analogous to the generating functions for involutions which are defined previously. However the only difference between the two groups of generating functions is that the method of constructing series (parallel) involutions under the transformation is identical to the method of constructing parallel (series) permutations σ. with $\sigma^h = \sigma^v$. The role of (y,z) and U_1 (y,z) in equations (3.4)-(3.8) must be interchanged to suit this class. This also means that $V_1(y,z) = U_3(y,z)$ and $U_1(y,z) = V_2(y,z)$.

From the trees illustrated in Fig.5 and Fig.6, it is clear that, the generating functions for permutations of Type (1) must be obtained to find the numbers of various types of permutations.

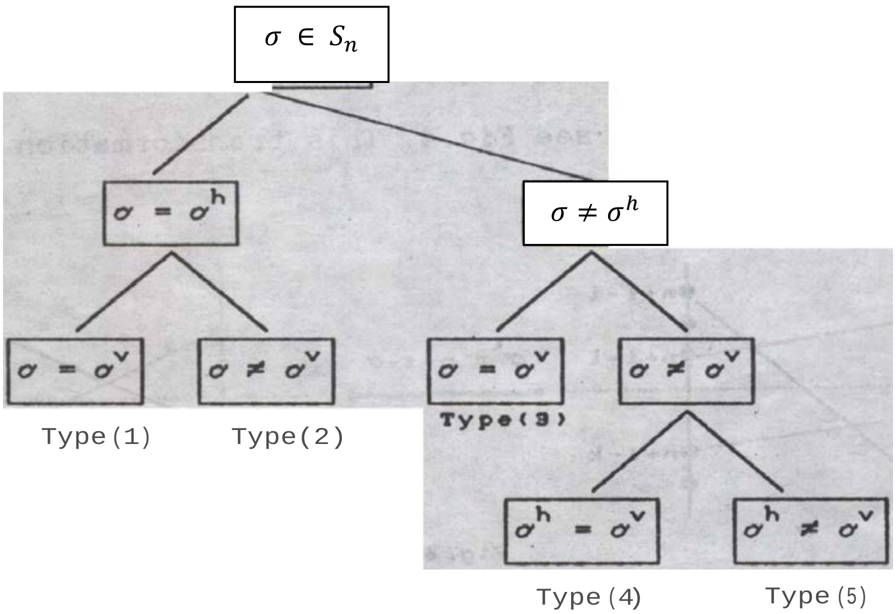


Fig.5: A distrubution tree of permutation w.r.t. flipping on horizontally and then on vertically.

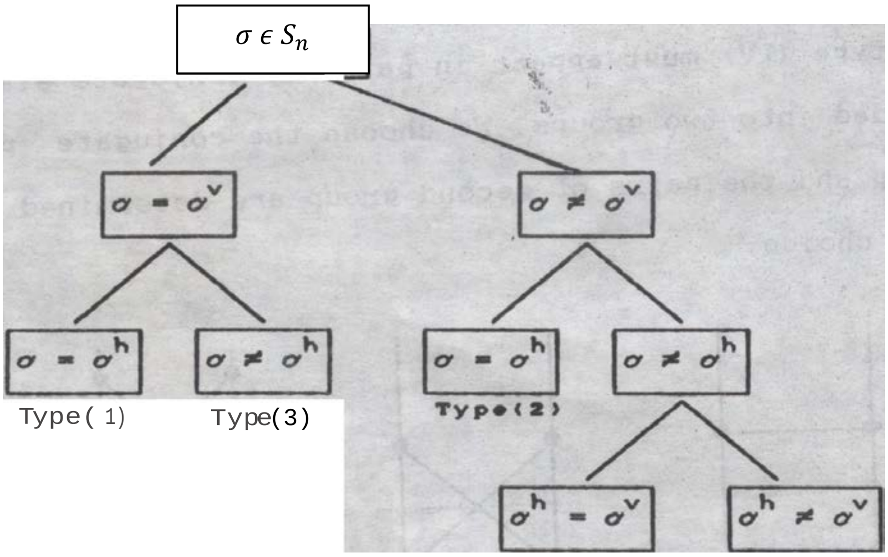


Fig.6: A distrubution tree of permutations w.r.t. flipping on vertically and then on horizontally.

The key to our enumeration depends on characterizing the elements of a permutation of this type. Elements of permutation $\sigma \in S_n$ of Type (1) must be partitioned into four types which are

Type (I) All elements $i \in [n]$ satisfy:

$\sigma(i) = i \wedge \sigma(n+1-i) = n+1-i$,

Type (II) All elements $k \in [n]$ satisfy:

$\sigma(k) = n+1-k \wedge \sigma(n+1-k) = k$,

Type (III) All pairs of distinct elements $\ell, k \in [n]$ satisfy:

$\sigma(k) = \iota \wedge \sigma(\iota) = k$ and $\sigma(n+1-k) = n+1-\iota \wedge \sigma(n+1-\iota) = n+1-k$,

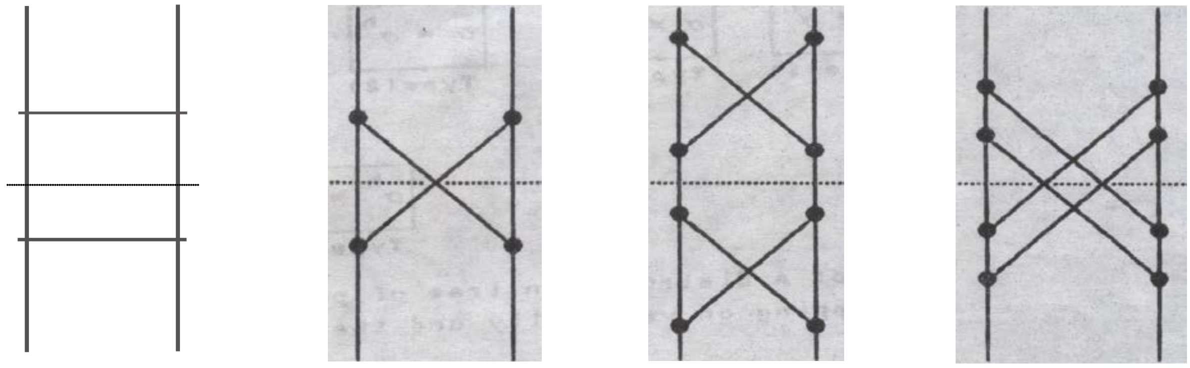
where $\ell, k, n+1-e$ and $n+1-k$ are pair wise distinct elements.

Type (IV) All pairs of distinct elements $\ell, k \in [n]$ satisfy:

$\sigma(\iota) = n+1-k \wedge \sigma(k) = n+1-\iota$ and $\sigma(n+1-k) = \iota \wedge \sigma(n+1-\iota) = k$,

where $\ell, k, n+1-e$ and $n+1-k$ are Pairwise distinct elements.

Schematic diagrams of these types are shown in Fig.7. It is clear, that the elements of type (I) and type (II) must appear in pairs of conjugate elements (2-cycle). Also the elements of type (III) and type (IV) must appear in pairs of conjugate elements which are divided into two groups. We choose the conjugate pairs of first group and the pairs of second group are determined according to this choice.



Type(I) Type(II) Type(III) Type (IV)

Fig. 7.

Now the generating functions for this class can be determined. Let $F(x,y,z,w) = \Sigma_{k,r,s,t\geq 0}\ f(k,r,s,t)(1+x)x^{2k}y^{2r}z^{2s}w^{2t}$ where $f(k,r,s,t)$ is the number of permutations $\sigma \in S_n$, $(k+r+s+t) = \lfloor n/2 \rfloor$, having k, r, s and t pairs of conjugate elements of type (I), type (II), type (III) and type (IV) respectively. Then $f(k,r,s,t)$ is the number of permutations having the eyele type $1^{2k}2^{r+s+t}$. Hence the coefficients of F(x,y,z,w) can be determined by

$$f(k,r,s,t) = \binom{m}{k}\binom{m-k}{r}\binom{m-k-r}{s}\frac{S!\qquad t!}{2^{\frac{s+t}{2}}\ \frac{s}{2!}\ t/2!}, \qquad (3.12)$$

where $m = \lfloor n/2 \rfloor$ and $t = m-(k+r+s)$.

Now we introduce further generating functions for this type.

$V(x,y,z,w) = \Sigma_{k,r,s,t\geq 0}\ V(k,r,s,t)(1+x)x^{2k}y^{2r}z^{2s}w^{2t}$,

$U(x,y,z,w) = \Sigma_{k,r,s,t\geq 0}\ U(k,r,s,t)(1+x)x^{2k}y^{2r}z^{2s}w^{2t}$,

$I(x,y,z,w) = \Sigma_{k,r,s,t\geq 0}\ I(k,r,s,t)(1+x)x^{2k}y^{2r}z^{2s}w^{2t}$,

$P(x,y,z,w) = \Sigma_{k,r,s,t\geq 0}\ P(k,r,s,t)(1+x)x^{2k}y^{2r}z^{2s}w^{2t}$,

where V(k,r,s,t), U(k,r,s,t), I(k,r,s,t) and P(k,r,s,t) are respectively the number of series, parallel, (+,⊕)-irreducible and prime permutations of cycle type $1^{2k}2^{r+s+t}$. Similarly to equation (3.5), we have

$$F(x,y,z,w) = V(x,y,z,w) + U(x,y,z,w) + I(x,y,z,w) \quad (3.13).$$

Lemma 3.4.

$$V(x,y,z,w) = (1+x)(F_1(x^2,z^2) - v_1(x^2,z^2))(1+F(x,y,z,w)).$$

Proof.

Let θ be a series permutation of Type (1) (i.e. $\theta = \theta^v = \theta^h = \theta^{vh}$).

Then θ can be written uniquely in either the form $\sigma \oplus \delta \oplus \emptyset$ or the form $\sigma \oplus \emptyset$. This leads to $(\sigma \oplus \delta \oplus \emptyset)^v = (\sigma \oplus \delta \oplus \emptyset)^h = (\sigma \oplus \delta \oplus \emptyset)^{vh}$, i.e. $(\emptyset^v \oplus \delta^v \oplus \sigma^v) = (\sigma^h \oplus \delta^h \oplus \emptyset^h) = (\emptyset^{vh} \oplus \delta^{vh} \oplus \sigma^{vh})$. Therefore, δ is any permutation having $\delta = \delta^h = \delta^v = \delta^{vh}$ and σ ($\emptyset = \sigma^v$) any non-series permutation in the class containing a permutation τ, having $\tau = \tau^h$, that can be enumerated by the term $F_1(x^2,z^2) - v_1(x^2,z^2)$.

The absence of δ is accounted for by the term 1 in the third bracket. The term x in the first bracket creates the odd terms of the series $V(x,y,z,w)$. □

Lemma 3.5.

$$U(x,y,z,w) = (1+x)\,(F_3(y^2,w^2) - U_3(y^2,w^2))(1+F(x,y,z,w)).$$

Proof. Similar to proof of lemma 3.4. □

Lemma 3.6. The generating function, $I(x,y,z,w)$, for $(+,\oplus)$-irreducible permutations of Type (1) is equal to

$$(1+F(x,y,z,w))\, P\{(F_1(x^2,z^2))^p, (F_3(y^2,w^2))^p, (F(z^4))^q, (F(w^4))^q\},$$

where p = 1/2 and q = 1/4.

Proof.

Let $\sigma \in S_n, \lfloor n/2 \rfloor = (k+r+s+t)$, and $\sigma = \sigma^h = \sigma^v = \sigma^{vh}$ be a prime permutation having k, r, s and t pairs of conjugate elements of type (I), type (II), type (III) and type (IV) respectively.

Consider any permutation θ ($\theta = \theta^v = \theta^h = \theta^{vh}$) obtained from by substituting the same permutation τ in the same elements of two related pairs of type (III) and type (IV). The pairs of conjugate elements of type (I) are replaced by the same permutation δ from a class having $\delta = \delta^h$. Also, the pairs of conjugate elements of type (II) are replaced by any permutation $\emptyset$ where $\emptyset^v = \emptyset^h$.
Therefore the generating function for all permutations obtained by substitution from σ is

$(F_1(x^2, z^2))^k (F_3(y^2, w^2))^r (F(z^4)^{s/2} (F(w^4))^{t/2}$. If the number, n, of elements is odd then there exists a middle point. This point must replace by a permutation from the class containing σ. Thus the generating function for all permutations obtained by substitution from σ with odd number is

$$(F(x,y,z,w))(F_1(x^2,z^2))^k(F_3(y^2,w^2))^r(F((z^4)^{s/2}(F(w^4))^{t/2}\ . \qquad \square$$

Equations (3.12) and (3.13) and lemmas 3.4, 3.5 and 3.6 can be used to calculate recursively V(x,y,z,w), U(x,y,z,w), I(x,y,z,w) and P(x,y,z,w).

§4. Permutation Graphs.

Now a functional equation for the generating function for prime permutation graphs, (or UTOs except the graphs with n ≤ 3), in terms of the generating functions for corresponding types of the prime permutations can be obtained as follows.

Theorem 4.1.
Consider $P^g(x) = \Sigma_{n>1} P^g(n) x^n$ be the generating function for prime permutation graphs. Then

$$P^g(X) = P_1^g(X) + P_2^g(X) + P_3^g(X),$$

where. (a) $P_1^g(x) = P(x,x,x,x)$.
(b) $P_2^g(x) = 1/2\ \Sigma_{i=1}^3 (P_i(x,x) - P(x,x,x,x))$.

(e) $P^g_s(x) = 1/4\ (P(x) - \Sigma^3_{i=1} P_i(x,x) + 2P(x,x,x,x))$.

Proof.

Let σ be a prime permutation. The graph G corresponding to σ is then prime (§2). Also, we know that $\Gamma(\sigma)$ is the set of all permutations corresponding to G. So we must divide the coefficients of generating functions for permutations by $|\Gamma(\sigma)|$. Obviously there exist three cases:

(a) P(x,x,x,x) is the generating function for permutations σ of Type (1), $(|\Gamma(\sigma)|) = 1)$.

(b) P_i(x,x)-P(x,x,x,x) is the generating function for permutations σ of Type (i), i= 2,3,4, $(|\Gamma(\sigma)|) = 2)$.

(c) P(x)- $\Sigma^3_{i=1} P_i$(x,x) + 2P(x,x,x,x) is the generating function for permutations σ of Type (5), i.e. $(|\Gamma(\sigma)|) = 4)$.

This completes the proof of theorem 4.1. ▭

Lemma 4.2

Let G be a prime permutation graph and let Ω(G) denote its automorphism group. Then $|\Omega(G)| \leq 4$.

Proof.

For any permutation σ, all elements of $\Gamma(\sigma)$ certainly represent PG(σ) say G. According to theorem 2.1 if any two or more elements of $\Gamma(\sigma) = \{\sigma, \sigma^v, \sigma^h, \sigma^{vh}\}$ are equal this means that there exists an automorphism of G gives this equivalent. Therefore, we have the following cases: (i) if $|\Gamma(\sigma)| = 4$ then there is only the trivial automorphism, id, i.e. $|\Omega(G)| = 1$. (ii) if $|\Gamma(\sigma)| = 1$ then there is four elements in Ω(G) that are {id, α, β, $\alpha o \beta$} where α(G) corresponding to σ^{vh}. β(G) corresponding to σ^v and $(\alpha o \beta)$ (G) Corresponding to σ^{vh}. (iii) if $\sigma = \sigma^h$ or $\sigma = \sigma^v$ or $\sigma = \sigma^{vh}$ then Ω(G) has respectively {id, α} or {id, β} or {id, $\alpha o \beta$}. This shows that Ω (G) has at most four elements. □

Let G — (V.E) be a prime permutation graph. Assume that G has a non-trivial automorphism group Ω (G). say $\Omega(G)=\{id,\tau_1,\tau_2,\tau_3\}$. $\tau_3 = \tau_1 \circ \tau_2$. A vertex v $\in$ V for which τ_i (v) – v • $\tau_i \in \Omega$ (G) will be called *fixed vertex*. A pair of distinct vertices v, u $\in$V for which $\tau_i(v)$ = u and τ_i (u) = v, $\tau_i \in \Omega(G)$ will be called *conjugate* vertices. Sometimes if Ω(G) has four elements and$(x_1,x_2).(x_3,x_4) \in \tau_i \wedge (x_1x_4, (x_2x_3) \in \tau_2$ then the composition of automorphisms τ_1, τ_2 produces 4-cycle. i.e. $(x_1x_2x_3x_4) \in \tau_1 \circ \tau_2$. In this case these two pairs of conjugate vertices will be called *conjugate tetrad vertices*. If G has the trivial automorphism group then, by convention, all vertices are considered to be fixed.

Let P^g (y,z) = $\Sigma_{j,k\geq 0} P^g(j,k) y^j z^{2k}$ where P^g (j,k) is the number of prime unlabeled permutation graphs having j fixed vertices and k pairs of conjugate vertices.

Consider also P^g (y, z,w) - $\Sigma_{j,k,r\geq 0}$ P (j,k,r) $y^j z^{2k} w^{4r}$ where P (j,k,r) is the number of prime unlabeled permutation graphs having j fixed vertices, k pairs of conjugate vertices and r conjugate tetrad vertices. We can re-state theorem 4.1 as follows:

Theorem 4.3.

The generating function for total prime permutation graphs is given by P^g (y,z,w) + P^g (y,z), where P^g (y,z,w) = P (y,z,w,w) and P^g (Y,z) • $1/2\, \Sigma_{i=1}^{3}(P_i$ (y,z)-P (y,z,z,z)) + 1/4 . $(P(y)-\Sigma_{i=1}^{3} P_i(y,y)$ + 2P (y,y,y,y) . □

This formula is the best to enable anyone to obtain the generating function $I^g(x) = \Sigma_{n\geq 1} i^g(n) x^n$ for(+,$\oplus$)-irreducible permutation graphs in terms of the generating function. G(x), for total permutation graphs .

Lemma 4. 4- •

$$I^g(X) = X + I_1^g(X) + I_2^g(X),$$

where

$$I_1^g(X) = 1/4 \sum_{j,k,r\geq 0} P(j,k,r)[1+G(X)][G^n(X) + G^j(X)G^{k+2r}(X^2) + G^{2k}(X)G^{\frac{j}{2}+2r}(X^2) + G^{n/2}(X^2)] .$$

and $P(j,k,r)$ are the coefficients of the generating function P^g (y,z,w) (theorem 4 .3) and n = j + 2k + 4r.

$$I_2^g(X) = 1/2 \sum_{j,k\geq 0} P^g(j,k)[G^{j+2k}(X) + G^j(X)G^k(x^2)] .$$

and P^g (j,k) are the coefficients of the generating function $P^g(y,z)$ (theorem 4 .3) and n = j +2 k .

Proof.

In the simple case, suppose G be a prime permutation graph with j fixed vertices and k pairs of conjugate vertices. Then the automorphism group, Ω(G), has the cycle index

$$Z(\Omega(G)) = 1/2(S_1^{j+2k} + S_1^j S_2^k).$$

In the other case. G has j fixed vertices. k pairs of conjugate vertices and r conjugate tetrad vertices . Then the automorphism group, Ω (G) . has the cycle index

$$Z(\Omega(G)) = \begin{cases} 1/4(S_1^n + S_1^j S_2^{k+2r} + S_1^{2k} S_2^{\frac{j}{2}+2r} + S_2^{\frac{n}{2}}, & n\ even \\ 1/4(S_1^n + S_1^{j+1} S_2^{k+2r} + S_1^{2k+1} S_2^{\frac{j}{2}+2r} + S_1 S_2^{\frac{n}{2}}, & n\ odd \end{cases}$$

$\lfloor n/2 \rfloor = j + k + r$. From Polya's enumeration theorem ([4], ch .3, p.35) .the generating function for all permutation graphs obtained fram G by substitution canposition is Z(Ω (G), $s_i \leftarrow G(x^i)$. The term "x" in the R.H.S. determines the single vertex graph that is considered (+,⊕)-irreducible but cannot be obtained by substitution fram a prime graph .

Let us introduce same more generating functions . Let $G(x)= \Sigma_{n\geq 1} g(n)x^n$, $C(x) =\Sigma_{n\geq 1} c(n) x^n$, $D(x) = \Sigma_{n\geq 1} d(n) x^n$, where g (n), c (n) , and d (n) denote the number of all, series , and parallel permutation graphs with n vertices . Some relations between these and $I^g (X)$ will be proved in the following proposition .

Proposition 4.5.

(a) $G(x) = C(x) + D(x) + I^g(x)$.

(b) $C(x) = D(x)$.

(c) $1 + G(x) = \exp \Sigma_k 1/k \ (C(x^k) + I^g(x^k))$.

Proof.

a) is clear . To prove (b) , we must take in our account that the *prime image* , the graph which is obtained by reducing every maximal non-trivial partitive set to a one vertex, in both cases is the complement of the other . Since the prime image of parallel permutation graphs with n vertices is the independent set of n vertices and in case of series permutation graphs .is the complete graph of order n . (c) is the direct application of Riddell's theorem (see[4] ch.4. -p .90)for determining the generating function for total graphs by knowing the generating function for connected graphs if the properties of graphs are hereditary. □

The above proposition completes the recurs1ve calculation of g (n) . The nun1ber of pennutation graphs with n vertices for n S 20 are included in the appendix .

Appendix.

The algorithms (recurrence relations) developed in this paper were programmed by the authors. The progra1ns were run on a PC/XT with 8 MHz 8088-1 processor and made use of double precision via 8087 math coprocessor .

A list of notations of what numbers are available is given below. Followed by tables of these numbers . The number of prime permutation graphs divided into two parts. The first part, P(j,k,r), is the number of prime permutation graphs that correspond to the number of permutations of Type (1). The second part. $P^g(j,k)$. is the number of prime permutation graphs that correspond to the half of total number of permutations of types (2)-(4) plus the quarter of the number of permutations of type (5).

P(j,k,r) :the number of prime unlabeled permutation graphs with j fixed vertices, k pairs of conjugate vertices and r conjugate tetrad vertices.

$P^g(j,k)$: the number of prime unlabeled permutation graphs with j fixed vertices and k pairs of conjugate vertices.

$P^g(n)$: the number of prime permutation graphs with **n** vertices.

$i^g(n)$:the number of (+,⊕)-irreducible permutation graphs with n vertices.

c(n) :the number of series permutation graphs with n vertices.

g(n) :the number of total permutation graphs with n vertices.

The values of P(j,k,r) ($P^g(j,k)$) for n≥20 where n = j+2k+4r. (n=j+2k) are given in Table(1) Table (2). Also, Table (3) contains the values of $P^g(n)$, $i^g(n)$, c(n) and g(n) for n ≤20.

Remark:

In table (1), the numbers of prime permutation graphs with n vertices are equal to the numbers of prime permutation graphs with n+1 vertices.

Table -1-

n = 6 :	
P(0,1,1) = 1	P(2,0,1) = 1
n = 8 :	
P (0,0,2)= 2	P (2,1,1)= 4
n = 10 :	
P (0,1,2)= 9	P(2,0,2) = 9
P(2,2,1) = 2	P(4,1,1) = 2
n = 12 :	
P (0,0,3)= 20	P(0,2,2) = 13
P(2,1,2) = 56	P(4,0,2) = 13
P(4,2,1) = 4	
n = 14 :	
P (0,1,3)= 140	P(0,3,2) = 5
P(2,0,3) = 140	P(2,2,2) =102
P(4,1,2) = 102	P(4,3,1) = 2
P(6,0,2) = 5	P(6,2,1) = 2
n = 16 :	
P (0,0,4)=306	P(0,2,3) =352
P(2,1,3) =1100	P(2,3,2) =76
P(4,0,3) =352	P(4,2,2) =256
P(6,1,2) = 76	P(6,3,1) = 4
n = 18 :	
P (0,1,4)=2699	P(0,3,3) =422
P(2,0,4) =2699	P(2,2,3) =3240
P(2,4,2) = 20	P(4,1,3) =3240
P(4,3,2) = 312	P(6,0,3) = 422
P(6,2,2) = 312	P(6,4,1) = 2
P(8,1,2) = 20	P(8,3,1) = 2
n = 20 :	
P (0,0,5)=5828	P(0,2,4) =9657
P(0,4,3) =240	P(2,1,4) =26168
P(2,3,3) =4920	P(4,0,4) =9657
P(4,2,3) =11504	P(4,4,2) =184
P(6,1,3) =4920	P(6,3,2) =616
P(8,0,3) =240	P(8,2,2) =184
P(8,4,1) =4	

Table -2-

$P^g(j,k)$

j \ k	0	2	3	4	5	6	7	8	9	10
0	0	1	4	48	546	7086	104276	1726414	31849302	648736066
1	0	3	12	130	1490	20050	308068	5328520	102498422	
2	0	2	24	340	5008	82548	1507680	30312044	665972928	
3	0	0	32	702	13920	286880	6283440	147184598		
4	0	0	18	808	23944	650172	17598648	489709830		
5	0	0	2	526	26680	1012340	35019768			
6	8	0	0	182	19240	1104452	50934512			
7	78	0	0	20	8632	844748				
8	694	0	0	0	2210	444202				
9	6954	0	0	0	226					
10	74174	0	0	0	0					
11	862506	0	0	0						
12	10815154	0	0	0						
13	145941638	0	0							
14	2108391296	0	0							
15	32484992348	0								
16	531835339072	0								
17	9222255576472									
18	168887116735932									
19	3257683203461876									
20	66027855230798860									

Table-3-

n	$P^g\ (n)$	$i^g\ (n)$	$c\ (n)$	$g\ (n)$
1	0	1	0	1
2	0	0	1	2
3	0	0	2	4
4	1	1	5	11
5	3	7	13	33
6	16	56	43	142
7	92	424	176	776
8	772	3807	946	5699
9	7122	37413	6655	50723
10	75100	409236	57668	524572
11	864722	4867950	584784	6037518
12	10828162	62620895	6645569	75912033
13	145976240	864411955	82781507	1029974969
14	2108602744	12743980444	1115117484	14974215412
15	32485614494	199839950926	16116330326	232072611578
16	531839245100	3321988198540	248522321145	3819032840830
17	9222268211926	58368315636178	4071974487560	66512264611298
18	168887197615978	1081152612514174	70646883155506	1222446378826186
19	3257683489023028	21062015593620296	1294039677219392	23650094948059080
20	66027857086670520	430599259434792202	24959331421832611	480517922278457424

References.

(1) M. Aginer and G. Prins, (1971), "Uniquely partially orderable "graphs". J. London Math. Soc., vol. 2, p. 11-28.

(2) B. I. Bayoumi, M. H. El-Zahar & S. M. Khamis. (1990). "Counting Two-Dimensional Posets". accepted (J. Disc. Math.).

(3) S. Even, A. Lempel and A. Pnueli. (1972), "Permutation graphs and transitive graphs", J. ACM. vol. 19, p. 400- 410.

(4) F. Harary & E. palmer, (1973), "Graphical Enumeration". New York. Landon; Academic Press.

(5) R. H. MOhring, (1984). "Almost all comparability graphs are UPO", J. Disc. Math, vol.50, p. 63-70.

(6) R. Mohring, (1985). "Algorithmic aspects of comparability graphs and interval graphs". in Ordered sets, p. 41-101, (I. Rival ed.), Reidel, Dordrecht.

(7) J. Spinrad, (1985), "On comparability and permutation graphs. SIAM J. Comput., vol. 14, p. 658-670.

(8) L. k.Stewart, (1985), "Permutation Graph Structure And Algorithms", Thesis, Dep. of Computer Science, University of Toronto, Canda.

جامعة القاهرة

معهد الدراسات والبحوث الإحصائية

المؤتمر السنوي الخامس والعشرون

للاحصاء وعلوم الحاسب والمعلومات

وبحوث العمليات

٨ - ١٠ ديسمبر ١٩٩٠